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Advent of Quaternions
Dedicated to 300th anniversary of Leonhard Euler's birth

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Abstract

The arrival of quadruples as an extension beyond the ordered pairs and so-called triples, may undoubtedly be considered as a remarkable achievement in the development of modern Mathematics. It resulted in the birth of quaternions as advances from complex numbers. In the present effort, discovery of quaternions, its algebraic aspect and the very role in the upcoming of abstract mathematics, are included. Applications of this modern concept in the emerging of new areas, are accommodated in brief.

Key words: Quaternions, Complex Numbers, Triples, Cardano, Euler, Hamilton, Algebras, Abstract Mathematics, Non-commutative Algebra.

Introduction

It is now known to us that complex numbers which are usually expressed as an ordered pair of reals, were introduced by the Italian mathematician Girolamo Cardano (1501-1576) as an extension over the real numbers (Stillwell, 2005). Cardano who was born at Pavia on September 24, 1501 and died at Rome on September 21, 1576, published his chief mathematical work *Ars Magna* in Nuremberg in 1545 (Rouseball, 1960). This is taken as a great advance of any algebra, earlier published. Hitherto, algebraists had confined their attention to those roots of equations which were positive. Apart from discussing negative and complex roots, Cardano proved that in case of cubic equations, the latter would always occur in pairs. Further, the theory of imaginary quantities received little attention from the mathematicians, until John Bernoulli and Euler (1707-1783) took up the matter after a lapse of two centuries (Bourbaki, 1999). It may be worthwhile to report that the symbol ***i*** was introduced by Euler in 1777, for denoting the square root of (-1) and the modern theory is chiefly based on his researchers. About imaginary numbers, great German mathematician Leibnitz has said, ***The imaginary number is a fine and wonderful recourse of the divine spirit, almost an amphibian between being and not being*** (Bayer and Merzback, 1989). Extensions beyond the complex numbers to the discovery of quaternions, consisting of quadruples of reals, were made in 19th century by Hamilton. This class of mathematical quantities, generally called quaternions that may be taken as a generalized form of a complex number (Clugston, 1998), is regarded to day as one of the significant contributions of famous Hamilton. In the field of real and complex numbers, product is always commutative whereas for quaternions, i.e., in specially defined ordered numbers, multiplication is not commutative. Further, prior to the arrival of quaternions, real and complex quantities, were the usual basis of algebra and in turn, there prevailed no idea about non-commutative algebra. Hamilton has shown that quaternions could take an important role in the development of non-commutative algebra in particular and abstract algebra in general (Hamilton, 1969).

Advances from Euler to Hamilton

In the beginning of 18th century, Leonhard Euler was born at Bale on April 15, 1707 and died at Petrograd on September 7, 1783. In a life span of 76 years, his fundamental works include a huge number of books, articles and research papers. It is really unfortunate to mention that at an age about 30, he lost his one sight and at about 60 losing another sight, he became blind, yet his creativity did not end. In modern mathematics, he first used symbols, e.g., ***i*** for imaginary quantity, **e** for base of Napierian logarithms and **f(x)** for function of x. He also introduced the current abbreviations for the trigonometric functions and, expressed their relation with exponential function and ***i***. Among the 18th century born famous mathematicians, three personalities comparable with Leonhard, are commonly referred to as Archimedes, Newton and Karl Friedrich Gauss. In his honour, the French mathematician Pierre Simon Laplace rightly said, ***Read Euler, read Euler, he is the master of all.*** In 1979, Gauss pointed out that every integral algebraical function of one variable can be expressed as a product of real linear or quadratic factors. Hence every algebraical equation has a root of the form **a + i b.** In the beginning of 19th century, Ireland's William Rowan Hamilton is born on 1805 in Dublin at midnight of 3-4 August and died on September 2, 1865. He is regarded as one of the greatest mathematicians of 19th century. He first answered that a complex number is an ordered pair (a, b) of real numbers and these pairs are added and multiplied according to the rules:

(a, b) + (c, d) = (a + c, b + d) and (a, b) x (c, d) = (a c – b d, a d + b c). Hamilton realized that multiplication of the pairs of real numbers, is indeed an important question in its own right. However, he was interested in the bigger question of multiplying triples, quadruples, and so on. In case of addition of triples, there is the vector addition:

(a, b, c) + (d, e, f) = (a + d, b + e. c + f)

which can be generalized to n-tuples for any n. However, he was tormented by the problem of generalizing the multiplication rule for triples and quadruples.

About Complex Numbers

We know that complex numbers which are typically expressed in the form z = x + i y or as an ordered pair (x, y) of reals where x, y ε **R,** z ε **C** and i = √(-1), were used by famous Cardano as an extension beyond the real numbers. One may thus identify **C** with $\mathbf{R}^2$ and the operation of addition in **C** coincides with that of vector addition in $\mathbf{R}^2$. However, the parametric form of a unit circle, is taken by the mathematical model:

x = cos t , y = sin t,

where x and y represent the co-ordinates of a moving point, placed at a unit distance apart from the axis of rotation and t is a parameter. A gradual advance in this regard, is our familiar Euler's formula, after the great Swiss mathematician Euler that includes trigonometric functions through the exponents of an imaginary number as follows:

e^{it} = cos t + i sin t and its conjugate

e^{-it} = cos t – i sin t.

The new aspect of **C** with $\mathbf{R}^2$ is that the operation of multiplication for complex numbers, differs from scalar and vector products in $\mathbf{R}^2$(Sudbery, 1979; Eriksson et al., 2004)**.**

Arrival of Quaternions

Even though a necessity for complex numbers was first noticed in the 16th century with the solution of cubic equations, in the early part of the 19th century, complex numbers became a hot topic for carrying out research. Irish mathematician Sir W R Hamilton (1805-1865) formulated the first modern exposition of complex numbers in 1833. At that time, a burning question in algebra was, *If a rule for multiplying two numbers together is known, what is about multiplying three numbers?* Almost over a decade, this simple question troubled Hamilton who was under pressure from within and outside, for finding a possible answer. Hamilton's letter to his son reflects his anxiety, *Early morning in the early part of 1843, on my coming down to breakfast, your brother William Edwin and yourself used to ask me, 'Well papa, can you multiply triplets?' Whereto I was always agreed to reply, with a sad shake of the head, No, I can only add and subtract them.* He felt that his ordered pairs could be treated as directed entities in the plane and naturally, he applied the idea to 3-D (three- dimensional) by moving from the binary complex number $x + i y$ to ordered number triples $x + y i + z j$ or (x, y, z). In the process, addition created no problem, but for long ten years, multiplication of n - tuples for $n > 2$, was his unsolved problem. Thus, after a long pursuation, Hamilton invented the quaternions in 1843, while attempting to introduce a product of vectors in $\mathbf{R}^3$, similar to that of complex numbers in **C**. Although, he obtained a solution finally but it was weird and very weird as it was 4-D. It is said the idea of quaternions came to him in 1843 while walking one day with his wife along the Royal Canal, on his way to a meeting of the Irish Academy. He realized from the flash of inspiration, his difficulty would be over by the use of quadruples $x + y i + z j + t k$ or (x, y, z, t), instead of triples (x, y, z) and if the commutative law for product was abandoned. He was so pleased with the discovery, he stopped his walk and curved the fundamental formula of quaternion algebra, $i^2 = j^2 = k^2 = i j k = - 1$, with a knife on a stone of Brougham Bridge in Dublin and that memorable day was 6th October, 1843.

Algebraic Aspect of Quaternions

Algebra supplies tool for performing manipulation. The set of complex numbers can be identified with the points in the plane $\mathbf{R}^2$ and Hamilton was looking for an analogous algebra for describing 3-D space. His effort was to find an algebraic system that would do for the space $\mathbf{R}^3$, the same thing as complex numbers play for the plane $\mathbf{R}^2$. In particular, he wanted to find a multiplication rule for triplets: $\mathbf{a} = a_1 i + a_2 j + a_3 k$ and $\mathbf{b} = b_1 i + b_2 j + b_3 k$ so that $|\mathbf{a}\ \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$, $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$. However , no such bilinear products exist (at least not over rationals) as

$3 \times 21 = 63 \neq n_1^2 + n_2^2 + n_3^2$ for any integers n_1, n_2, n_3 though
$3 = 1^2 + 1^2 + 1^2$ and $21 = 1^2 + 2^2 + 4^2$ (no integer of the form

$4^a (8 b+7)$, with $a\geq 0$, $b\geq 0$), is a sum of three squares, a result of Legendre in 1830, $63 = 4^0 (8 \times 7 + 7)$. He also attempted in finding a generalized complex number system in 3-D, but no such associative hyper-complex numbers exist in 3-D. This can be seen by considering generalized imaginary units i and j such that $i^2 = j^2 = -1$ and such that

$\{1, i, j\}$ span $\mathbf{R}^3$. The product must be of the form $i j = \alpha + \beta i + \gamma j$ for some reals α, β, γ. Then $i(i j) = (\gamma \alpha - \beta) + i (\alpha + \beta \gamma) + j \gamma^2$, whereas by associativity $i(i j) = i^2 j = - j$ which leads to a contradiction since $\gamma^2 \geq 0$, for all real γ. Hamilton's great idea was to go to 4-D and to consider elements of the form $q = t + i x + j y + k z$ where the hyper - complex units i, j, k satisfy the following non-commutative multiplication rules for the basis $\{1, i, j, k\}$:

$i^2 = j^2 = k^2 = - 1$, $i j = k = - j i$, $j k = i = - k j$, $k i = j = - i k$,

which can be condensed into the form :

$i^2 = j^2 = k^2 = i j k = - 1.$

Hamilton named his 4-component element as quatenions which form a division ring, usually denoted by **H**, in honour of its inventor Hamilton (Lounesto, 1997; Altman, 1986).

Its Role in Modern Mathematics

In the opinion of some authors, the theory of quaternions will be ultimately esteemed as one of the great discoveries of the 19th century. The arrival of quadruples had opened a new horizon in the advancement of pure mathematics. Also, being inspired by Hamilton's modern ideas, Abstract Mathematics is developed by the famous mathematicians, in the 19th century. This newly formulated abstract mathematics have already accommodated Set Theory, Groups, Rings, Integral Domains, Fields, Vector Spaces, Lattices, Boolean Rings and Boolean Algebras, Jordan Algebras, Lie Algebras, Octonions etc.. The concept of Matrix in mathematics, came from the concept of quarternions (Trapp, 2005; Porteous, 1981).

Discussions

Indeed, it was the prolong search and study of rotation groups (3-D) that drove Hamilton toward the discovery of quaternions (4-D) in 1843. He treated quaternions as vectors and essentially showed that they form a linear vector space over the real number field. Further, quaternions are proved to be fundamental in several areas of mathematics and physics. In his book, ***Lectures***

On Quaternions in 1853, the idea of scalar and vector multiplications, has been referred and besides this, scalar and vector products are discussed. With the discovery of quaternions, there began a new trend of concept in modern mathematics. In quantity based algebra, commutativity in the operations of addition and multiplication, had been accepted, but the advent of quaternions has announced the presence and the proposed expansion in the non-commutative algebra.

Conclusion

Undoubtedly, it is true that Hamilton spent too much time on quaternions. It is really interesting to study the advances from the ordered pair (a, b) of real numbers to the arithmetic of triples (a, b, c) and quadruples (a, b. c, d). The discovery of quaternions by W R Hamilton in 1843, is regarded as a milestone in the history of mathematics. Prior to the arrival of this modern concept, there was no idea of non-commutative algebra. Specially, the idea of matrix came to exist out of quaternions. It has certainly opened a new horizon for modern concepts and logical developments, in different branches of science. It may be worthwhile in reporting that subsequently this discovery of quaternions was generalized by W C Clifford. In 1876, he had introduced the algebras bearing his name and now popularly called Clifford Algebra. Subsequently, Lie Algebra was developed by the great analyst Marius Sophus Lie of 19th century. It is worth mentioning that Lie algebras are not only non-commutative but also non-associative. Although, in application side, quaternions could not achieve much popularity because of its inherent complexity, however, these numbers have applications in Robotics, Euler angles and their unification, Octonions etc..

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